A Rigorous Treatment of Energy Extraction from a Rotating Black Hole

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Abstract

The Cauchy problem is considered for the scalar wave equation in the Kerr geometry. We prove that by choosing a suitable wave packet as initial data, one can extract energy from the black hole, thereby putting supperradiance, the wave analogue of the Penrose process, into a rigorous mathematical framework. We quantify the maximal energy gain. We also estimate the infinitesimal change of mass and angular momentum of the black hole, in agreement with Christodoulou's result for the Penrose process. The main mathematical tool is our previously derived integral representation of the wave propagator.

1 Introduction and Statement of Results

Near a rotating black hole there is the counter-intuitive effect that the physical energy of particles or waves need not be positive. As discovered by Roger Penrose [10], this effect can be used to extract energy from the black hole. In the process proposed by Penrose, a classical particle enters the so-called *ergosphere*, a region outside the event horizon, where it disintegrates into two particles. One particle of negative energy falls into the black hole, whereas the other particle has energy greater than the initial energy, and escapes to infinity. The wave analogue of the Penrose process is called *superradiance*; it was proposed in [14] and first studied in [11, 12], see also [2, 9, 13]. In these papers, superradiance was considered only on the level of modes, i.e., by considering the reflection coefficients for the radial ODE obtained after separating out the time and angular dependence. A more convincing treatment of superradiance is to consider the Cauchy problem for "wave packet" initial data, and in [1], this was carried out numerically for the scalar wave equation. In this paper we shall consider the Cauchy problem analytically, giving a mathematically rigorous treatment of superradiance for scalar waves. Our analysis is based on the integral representation for the wave propagator obtained in [6, 7].

A rotating black hole is modeled by the Kerr metric, which in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, takes the form

$$\begin{aligned} ds^2 &= g_{jk}\, dx^j\, dx^k \\ &= \frac{\Delta}{U}\,(dt - a\,\sin^2\vartheta\, d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{U}\,(a\,dt - (r^2+a^2)\,d\varphi)^2. \quad (1.1) \end{aligned}$$

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Here $M > 0$ and $aM > 0$ denote the mass and the angular momentum of the black hole, respectively, and the functions U and Δ are given by

$$U(r, \vartheta) \;=\; r^2 + a^2 \cos^2\vartheta \;, \qquad\qquad \Delta(r) \;=\; r^2 - 2Mr + a^2 \;.$$

We consider only the *non-extreme case* $M^2 > a^2$, where the function Δ has two distinct roots. The largest root

$$r_1 \;=\; M \;+\; \sqrt{M^2 - a^2}$$

defines the the event horizon of the black hole. We restrict attention to the region $r > r_1$ outside the event horizon where $\Delta > 0$. The metric (1.1) does not depend on t nor φ and is thus stationary and axisymmetric, admitting the two commuting Killing fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$. The *ergosphere* is defined to be the region where the Killing field $\frac{\partial}{\partial t}$ is space-like, that is where

$$r^2 - 2Mr + a^2 \cos^2\vartheta < 0 \,. \tag{1.2}$$

The ergosphere lies outside the event horizon $r = r_1$, and its boundary intersects the event horizon at the poles $\vartheta = 0, \, \pi$.

We now briefly review the Penrose process (for more details see [13]). The 4-momentum p^j of a massive point particle is time-like and future-directed, and thus its energy $\langle p, \frac{\partial}{\partial t} \rangle$ is positive whenever $\frac{\partial}{\partial t}$ is time-like. However, in the ergosphere the vector $\frac{\partial}{\partial t}$ becomes space-like, and hence the energy of the point particle need *not* be positive. Penrose considers a particle of positive energy which splits inside the ergosphere into two particles whose energies have opposite signs. By finely tuning the energy and momenta of these particles, one can arrange that the particle of negative energy crosses the event horizon and reduces the angular momentum of the black hole, whereas the other particle escapes to infinity, carrying (due to energy conservation) more energy than the original particle. In this way, one can extract energy from the black hole, at the cost of reducing its angular momentum. Christodoulou [3] showed that the infinitesimal changes of mass δM and angular momentum $\delta(aM)$ of the black hole satisfy the inequalities

$$\delta(aM) \;\leq\; \frac{r_1^2 + a^2}{a}\, \delta M \;<\; 0 \,, \tag{1.3}$$

and as a consequence he showed that it is not possible to reduce the mass of the black hole via the Penrose process below the *irreducible mass* M_{irr} defined by

$$M_{\text{irr}}^2 \;=\; \frac{1}{2} \left(M^2 + \sqrt{M^4 - (aM)^2} \right) \,. \tag{1.4}$$

We now recall the wave equation in the Kerr geometry and its separation; for more details see [6]. The scalar wave equation is

$$g^{ij} \nabla_i \nabla_j \, \Phi = \frac{1}{\sqrt{-\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{-\det g} \, g^{ij} \frac{\partial}{\partial x^j} \right) \Phi \;=\; 0 \,, \tag{1.5}$$

and in Boyer-Lindquist coordinates this becomes

$$\begin{aligned} \Bigg[-\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} + \frac{1}{\Delta} \left((r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} \right)^2 - \frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial \cos\vartheta} & \\ - \frac{1}{\sin^2\vartheta} \left(a \sin^2\vartheta \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right)^2 \Bigg] \Phi \;=\; 0 \,. & \end{aligned} \tag{1.6}$$

Using the ansatz

$$\Phi(t,r,\vartheta,\varphi) = e^{-i\omega t - ik\varphi}\, R(r)\, \Theta(\vartheta) \tag{1.7}$$

with $\omega \in \mathbb{R}$ and $k \in \mathbb{Z}$, the wave equation can be separated into both angular and radial ODEs,

$$\mathcal{R}_{\omega,k}\, R_\lambda \;=\; -\lambda\, R_\lambda , \qquad \mathcal{A}_{\omega,k}\, \Theta_\lambda \;=\; \lambda\, \Theta_\lambda\,. \tag{1.8}$$

The angular operator $\mathcal{A}_{\omega,k}$ is also called the *spheroidal wave operator*. It has a purely discrete spectrum of non-degenerate eigenvalues $0 \leq \lambda_1 < \lambda_2, \ldots$. The corresponding eigenfunctions $\Theta_n^{\omega,k}$ are referred to as spheroidal wave functions. In order to bring the radial equation into a convenient form, we introduce a new radial function $\phi(r)$ by

$$\phi(r) \;=\; \sqrt{r^2+a^2}\; R(r)\,, \tag{1.9}$$

and define the Regge-Wheeler variable $u \in \mathbb{R}$ by

$$\frac{du}{dr} \;=\; \frac{r^2+a^2}{\Delta}\,, \tag{1.10}$$

mapping the event horizon to $u = -\infty$. The radial equation then takes the form of the Schrödinger equation

$$\left(-\frac{d^2}{du^2} + V(u) \right) \phi(u) \;=\; 0 \tag{1.11}$$

with the potential

$$V(u) \;=\; -\left(\omega + \frac{ak}{r^2+a^2} \right)^2 \;+\; \frac{\lambda_n(\omega)\, \Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\; \partial_u^2 \sqrt{r^2+a^2}\,. \tag{1.12}$$

Starobinsky [11] analyzes superradiance on the level of modes. Using the notation in [7], we can explain his ideas and results as follows. We fix the separation constants $k > 0$, ω and λ_n. Introducing the notation

$$\Omega \;=\; \omega - \omega_0 \qquad \text{with} \qquad \omega_0 \;=\; -\frac{ak}{r_1^2+a^2}\,, \tag{1.13}$$

the potential (1.12) has the asymptotics

$$\lim_{u \to -\infty} V(u) \;=\; -\Omega^2\,, \qquad \lim_{u \to \infty} V(u) \;=\; -\omega^2\,.$$

Thus there are fundamental solutions $\acute{\phi}$ and $\grave{\phi}$ of (1.11) which behave asymptotically like plane waves,

$$\lim_{u \to -\infty} e^{-i\Omega u}\, \acute{\phi}(u) \;=\; 1\,, \qquad \lim_{u \to -\infty} \left(e^{-i\Omega u}\, \acute{\phi}(u) \right)' \;=\; 0 \tag{1.14}$$

$$\lim_{u \to \infty} e^{i\omega u}\, \grave{\phi}(u) \;=\; 1\,, \qquad \lim_{u \to \infty} \left(e^{i\omega u}\, \grave{\phi}(u) \right)' \;=\; 0\,, \tag{1.15}$$

(see [7] for details). The solution $\phi = \overline{\acute{\phi}}$ has, near the event horizon $u = -\infty$, the asymptotics $\phi(u) = e^{-i\Omega u}$. Taking into account the factor $e^{-i\omega t}$ in the separation, this corresponds to a plane wave entering the black hole. A general solution ϕ can be expressed as a linear combination of $\grave{\phi}$ and its complex conjugate,

$$\phi(u) \;=\; A\, \grave{\phi}(u) + B \overline{\grave{\phi}}(u)\,. \tag{1.16}$$

Equation (1.15) shows that the solution $\grave{\phi}$ behaves near infinity like $\grave{\phi}(u) = e^{-i\omega u}$. Hence the corresponding time-dependent solution behaves like the plane wave $e^{-i\omega(t+u)}$ and corresponds to an *incoming wave* propagating from infinity. Likewise, the solution $\overline{\grave{\phi}}$ corresponds to an *outgoing wave* propagating towards infinity. Computing the Wronskian of ϕ and $\overline{\phi}$ near the event horizon and near infinity gives the relation

$$|A|^2 - |B|^2 \;=\; \frac{\Omega}{\omega}\,. \tag{1.17}$$

The quantities $\omega^2|A|^2$ and $\omega^2|B|^2$ have the interpretations as the energy flux of the incoming and outgoing waves, respectively. If the right side of (1.17) is positive, the outgoing flux is smaller than the incoming flux, and this corresponds to ordinary scattering. However, if the right side of (1.17) is negative, then the outgoing flux is larger than the incoming flux; this is termed *superradiant scattering*. According to (1.17), superradiant scattering appears precisely when ω and Ω have opposite signs. Using (1.13), we see that superradiant scattering occurs if and only if ω lies in one of the following intervals

$$\omega_0 \;<\; \omega \;<\; 0\,, \qquad\quad 0 \;<\; \omega \;<\; \omega_0\,, \tag{1.18}$$

depending on the sign of ω_0. The gain in energy is determined by the quotient of outgoing and incoming flux,

$$\mathfrak{R} \;=\; \frac{|B|^2}{|A|^2}\,. \tag{1.19}$$

Starobinsky [11] made an asymptotic expansion for $\mathfrak{R}$ and found a relative gain of energy of about 5% for $k = 1$ and less than 1% for $k \geq 2$. This was verified later numerically in [1]. Teukolsky and Press [12] made a similar mode analysis for higher spin and found numerically an energy gain of at most 4.4% for Maxwell ($s = 1$) and up to 138% for gravitational waves ($s = 2$).

Our main result makes the above mode argument for the scalar wave equation mathematically precise in the setting of the Cauchy problem. To state our result, we combine Φ and $\partial_t\Phi$, as in [6], into a two-component vector $\Psi = (\Phi, i\partial_t\Phi)$. The physical energy of the wave is then given by $<\!\Psi, \Psi\!>$, where $<\!.,.\!>$ is the *energy scalar product*

$$\begin{aligned} <\!\Psi, \Psi'\!> \;=\; & \int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \left\{ \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) \overline{\partial_t \Phi}\, \partial_t \Phi' \right. \\ & \left. +\Delta\, \overline{\partial_r \Phi}\, \partial_r \Phi' + \sin^2\vartheta\, \overline{\partial_{\cos\vartheta} \Phi}\, \partial_{\cos\vartheta} \Phi' + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) \overline{\partial_\varphi \Phi}\, \partial_\varphi \Phi' \right\}. \end{aligned} \tag{1.20}$$

The energy radiated to infinity is defined by

$$E_{\text{out}} \;=\; \lim_{t\to\infty} <\!\Psi(t), \chi_{(2r_1,\infty)}(r)\, \Psi(t)\!>\,, \tag{1.21}$$

where χ is the characteristic function (note that, according to the pointwise decay result in [7], we could replace the argument $2r_1$ by any radius greater than r_1). We can now state our main theorem.

Theorem 1.1 *Consider the Cauchy problem for the scalar wave equation in the non-extreme Kerr geometry for initial data with compact support outside the event horizon,*

$$\Psi_0 \;=\; (\Phi_0, i\partial_t\Phi_0) \;\in\; C^\infty_0((r_1,\infty)\times S^2)^2\,,$$

having energy $<\Psi_0, \Psi_0>$. *Fix* $k > 0$, $n \in \mathbb{N}$ *and* $\omega \in (\omega_0, 0)$. *Then for any* $R > r_1$ *and* $\delta > 0$ *there is initial data* $\Psi_0 \in C_0^\infty((R,\infty) \times S^2)^2$ *such that the limit in (1.21) exists and*

$$\left| \frac{E_{\rm out}}{<\Psi_0, \Psi_0>} - \mathfrak{R} \right| \;\leq\; \delta$$

with $\mathfrak{R}$ *as in (1.19). The same inequality holds for* $k < 0$ *and* $\omega \in (0, \omega_0)$.

We emphasize that we allow the initial data to be supported arbitrarily far away from the event horizon. This is important in order to avoid artificial initial data which would not correspond to an energy extraction mechanism. For example, if one allows the support of the initial data to intersect the ergosphere, one could take initial data with zero total energy, in which case the quotient $E_{\rm out}/<\Psi_0, \Psi_0>$ could be made arbitrarily large.

We also compute the energy $E_{\rm bh}$ and the angular momentum $A_{\rm bh}$ of the wave component crossing the event horizon of the black hole to obtain the following theorem.

Theorem 1.2 *For any initial data* $\Psi_0 \in C_0^\infty((r_1,\infty) \times S^2)^2$, *the quantities* $E_{\rm bh}$ *and* $A_{\rm bh}$ *defined by (2.8, 5.1) satisfy the inequality*

$$A_{\rm bh} \;\leq\; \frac{r_1^2 + a^2}{a}\, E_{\rm bh}\,.$$

This shows explicitly that Christodoulou's estimates (1.3, 1.4) also hold for energy extraction by scalar waves, in agreement with Hawking's area theorem [13].

2 Absorbtion of Energy by the Black Hole

Recall from [7] that given initial data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$, the solution of the Cauchy problem for the scalar wave equation (1.6) can be represented as

$$\Psi(t,r,\vartheta,\varphi) \;=\; \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{-ik\varphi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n}\, \Psi^a_{k\omega n}(r,\vartheta)\, <\Psi^b_{k\omega n}, \Psi_0>\,, \quad (2.1)$$

where the sums and the integrals converge in $L^2_{\rm loc}$. In the above integral representation, the coefficients t_{ab} are defined by

$$t_{11} \;=\; 1 + {\rm Re}\,\frac{\alpha}{\beta}\,, \qquad t_{12} \;=\; t_{21} \;=\; -{\rm Im}\,\frac{\alpha}{\beta}\,, \qquad t_{22} \;=\; 1 - {\rm Re}\,\frac{\alpha}{\beta}\,, \qquad (2.2)$$

and the complex coefficients α and β are defined by

$$\grave{\phi} \;=\; \alpha\, \acute{\phi} + \beta\, \overline{\acute{\phi}}\,, \qquad (2.3)$$

where $\acute{\phi}(u)$ and $\grave{\phi}(u)$ are the fundamental solutions of the radial equation having the asymptotics (1.14) and (1.15), respectively. We will refer to the complex coefficients α and β as *transmission coefficients*. Finally the functions $\Psi^a_{k\omega n}(r,\vartheta)$, $a = 1,2$ are the solutions of the wave equation (1.6), with fixed quantum numbers k, ω, n, corresponding to the real-valued fundamental solutions of the radial equation given by

$$\phi^1 = {\rm Re}\,\acute{\phi}\,, \qquad \phi^2 = {\rm Im}\,\acute{\phi}\,. \qquad (2.4)$$

Here we shall always consider a fixed k-mode, and without loss of generality we assume that $k > 0$. For notational simplicity, from now on we omit the index k and the φ-dependence. Furthermore, we introduce the vector notation

$$\mathbf{\Psi}^{\omega n}(r,\vartheta) \;:=\; \begin{pmatrix} \Psi^1_{\omega n}(r,\vartheta) \\ \Psi^2_{\omega n}(r,\vartheta) \end{pmatrix}, \quad \mathbf{\Psi}^{\omega n}_0 \;:=\; \begin{pmatrix} <\Psi^1_{\omega n}, \Psi_0> \\ <\Psi^2_{\omega n}, \Psi_0> \end{pmatrix}, \qquad \mathbf{T} \;=\; (t_{ab})_{a,b=1,2}\,, \tag{2.5}$$

which allows us to write the integral representation (2.1) in the compact form

$$\Psi(t,r,\vartheta) \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t} \langle \mathbf{\Psi}^{\omega n}(r,\vartheta),\, \mathbf{T}\,\mathbf{\Psi}^{\omega n}_0 \rangle_{\mathbb{C}^2}\,. \tag{2.6}$$

We now introduce some basic quantities needed for the formulation of the superradiance property for wave packets. The *total energy* E_{tot} of an initial data $\Psi_0 \in C^\infty_0(\mathbb{R}\times S^2)^2$ for the scalar wave equation (1.6) is defined by

$$E_{\text{tot}} \;=\; <\Psi_0, \Psi_0>\,. \tag{2.7}$$

This energy is conserved throughout the time evolution of the scalar wave, meaning that if $\Psi(t)$ denotes the solution of the Cauchy problem for the scalar wave equation, then

$$<\Psi_0, \Psi_0> \;=\; <\Psi(t), \Psi(t)>\,.$$

Recall that the *outgoing energy*, which represents the energy radiated to infinity, has been defined in (1.21). Finally, the energy absorbed by the black hole is defined by

$$E_{\text{bh}} \;=\; \lim_{t\to\infty} <\Psi(t), \chi_{(-\infty,2r_1)}(u)\,\Psi(t)>\,. \tag{2.8}$$

We next derive useful expressions for these quantities in terms of the initial data and the transmission coefficients α and β. The expression for E_{tot} is an immediate consequence of (2.6).

Proposition 2.1 *Choose a fixed $k \in \mathbb{Z}$ such that $\omega_0 < 0$. Then*

$$E_{\text{tot}} \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, \langle \mathbf{\Psi}^{\omega n}_0, \mathbf{T}\,\mathbf{\Psi}^{\omega n}_0 \rangle_{\mathbb{C}^2}.$$

We next want to compute E_{bh} and E_{out}. First, we need an argument which ensures that the sum over the angular momentum modes converges. Since $E_{\text{out}} = E_{\text{tot}} - E_{\text{bh}}$ and the convergence of the angular sum is already known for the total energy according to Proposition 2.1, it suffices to consider for example E_{out}. In [8] it is shown that the outgoing energy is bounded uniformly in time, i.e.

$$<\Psi(t), \chi_{(2r_1,\infty)}(r)\,\Psi(t)> \;\leq\; C(\Psi) \qquad \text{for all } t\,.$$

Moreover, in [8] it is shown that the outgoing wave can be approximated by a finite number of angular momentum modes uniformly in time, i.e. for any $\delta > 0$ there is N such that

$$<\Psi^N(t), \chi_{(2r_1,\infty)}(r)\,\Psi^N(t)> \;\leq\; \delta \qquad \text{for all } t\,,$$

where

$$\Psi^N(t,r,\vartheta,\varphi) \;=\; \frac{1}{2\pi} \sum_{n\geq N} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t} \sum_{a,b=1}^{2} t^{\omega n}_{ab}\, \Psi^a_{\omega n}(r,\vartheta)\, <\Psi^b_{\omega n}, \Psi_0>\,.$$

This estimate will always allow us in what follows to interchange the limit $t \to \infty$ with the sum over the angular momentum modes.

To compute $E_{\rm bh}$, we make use of the following lemma, which is stated and proved in [5, Lemma 9.1].

Lemma 2.2 *For any Schwartz function $f \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$, let $A_\pm$ be defined by*

$$A_\pm \;=\; \lim_{t\to\infty} \int_{-\infty}^{u_0} du \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega-\omega')(t\pm u)} f(\omega,\omega')\,.$$

Then

$$A_+ \;=\; 2\pi \int_{-\infty}^{\infty} f(\omega,\omega) d\omega \qquad \textit{and} \qquad A_- \;=\; 0\,.$$

Proposition 2.3 *Choose a fixed integer $k > 0$. Then*

$$E_{\rm bh} \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, \langle \mathbf{\Psi}_0^{\omega n}, \mathbf{TPT}\, \mathbf{\Psi}_0^{\omega n}\rangle_{\mathbb{C}^2}\,,$$

where $\mathbf{P}$ is the matrix

$$\mathbf{P} \;=\; \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Proof. Substituting the integral representation (2.1) into (2.8), we find

$$E_{\rm bh} \;=\; \lim_{t\to\infty} \frac{1}{(2\pi)^2} \int_{r_1}^{2r_1} dr \int_{-\infty}^{\infty} \frac{d\omega'}{\omega'\Omega'} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i(\omega-\omega')t}\, \mathcal{E}^{\omega',\omega}(r)\,, \tag{2.9}$$

where

$$\mathcal{E}^{\omega',\omega}(r) \;=\; \sum_{n,n'\in\mathbb{N}} \sum_{a,b,c,d=1}^{2} t_{ab}^{\omega' n'} t_{cd}^{\omega n} <\Psi_0, \Psi_a^{\omega' n'}> <\Psi_d^{\omega n}, \Psi_0> \int_{S^2} \mathcal{E}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})\,,$$

and $\mathcal{E}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})$ denotes the bilinear form corresponding to the energy density as given in [6, Eq. (2.13)]. Near the event horizon, we can expand $\mathcal{E}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})$ to obtain

$$\begin{aligned}
&\frac{\Delta}{r^2+a^2}\, \mathcal{E}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n}) \\
&= \;(r_1^2+a^2) \left(\omega'\omega\, \overline{\Phi_b^{\omega' n'}}\, \Phi_c^{\omega n} + \overline{\partial_u \Phi_b^{\omega' n'}}\, \partial_u \Phi_c^{\omega n} \right) \;-\; \frac{a^2 k^2}{r_1^2+a^2}\, \overline{\Phi_b^{\omega' n'}}\, \Phi_c^{\omega n} \;+\; \mathcal{O}(e^{\gamma u}) \\
&= \;\Theta_{n'\omega'} \Theta_{n\omega} \left(\omega'\omega\, \overline{\phi_b^{\omega' n'}}\, \phi_c^{\omega n} + \overline{\partial_u \phi_b^{\omega' n'}}\, \partial_u \phi_c^{\omega n} \;-\; \frac{a^2 k^2}{(r_1^2+a^2)^2}\, \overline{\phi_b^{\omega' n'}}\, \phi_c^{\omega n} \right) \;+\; \mathcal{O}(e^{\gamma u})
\end{aligned}$$

with the constant γ as in [7, Eq. (3.9)]. Using this formula in (2.9) and writing the radial integral in terms of the Regge-Wheeler variable u, the factor $\Delta/(r^2+a^2)$ drops out. After taking into account the asymptotics

$$\phi_1^{\omega n}(u) \;=\; \cos(\Omega u) \;+\; \mathcal{O}(e^{\gamma u})\,, \quad \phi_2^{\omega n}(u) \;=\; \sin(\Omega u) \;+\; \mathcal{O}(e^{\gamma u})\,,$$

we can first apply Lemma 2.2 and then use the orthogonality of the spheroidal wave functions to obtain

$$E_{\rm bh} \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2\Omega^2} \sum_{n\in\mathbb{N}} \langle \mathbf{\Psi}_0^{\omega n}, \frac{\mathbf{TPT}}{2}\, \mathbf{\Psi}_0^{\omega n}\rangle_{\mathbb{C}^2} \left(\omega^2 + \Omega^2 - \frac{a^2 k^2}{(r_1^2+a^2)^2} \right).$$

We finally apply the identity

$$\omega^2 + \Omega^2 - \frac{a^2 k^2}{(r_1^2 + a^2)^2} \;=\; \omega^2 + \Omega^2 - \omega_0^2 \;=\; \omega^2 + \omega^2 - \omega\omega_0 \;=\; 2\omega\Omega\,. \qquad \blacksquare$$

Using the matrix identity

$$\mathbf{T} - \mathbf{TPT} \;=\; \mathbf{TQT} \quad \text{with} \quad \mathbf{Q} := \mathbf{T}^{-1} - \mathbf{P}\,,$$

a short calculation yields the following result.

Proposition 2.4 *Choose a fixed $k \in \mathbb{Z}$ such that $\omega_0 < 0$. Then*

$$E_{\text{out}} \;=\; \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega} \langle \mathbf{\Psi}_0^{\omega n}, \mathbf{TQT}\, \mathbf{\Psi}_0^{\omega n} \rangle_{\mathbb{C}^2}\,,$$

where $\mathbf{Q}$ is the matrix

$$\mathbf{Q} \;=\; \frac{\Omega}{2\omega} \begin{pmatrix} |\alpha - \beta|^2 & i(\alpha+\beta)\,\overline{(\alpha-\beta)} \\ -i(\alpha-\beta)\,\overline{(\alpha+\beta)} & |\alpha+\beta|^2 \end{pmatrix}.$$

3 Energy Propagation of Wave Packets near Infinity

We fix $\tilde{\omega} \in (\omega_0, 0)$ and $\tilde{n} \in \mathbb{N}$ and consider initial data Ψ_0 in the form of a linear combination of outgoing and incoming wave packets,

$$\Psi_0 \;=\; \begin{pmatrix} \Phi_0 \\ i\partial_t \Phi_0 \end{pmatrix} \;=\; \Theta_{\tilde{n},\tilde{\omega}}(\vartheta)\, \frac{\eta_L(u)}{\sqrt{r^2+a^2}} \left[c_{\text{in}}\, e^{-i\tilde{\omega}u} \begin{pmatrix} 1 \\ \tilde{\omega} \end{pmatrix} + c_{\text{out}}\, e^{i\tilde{\omega}u} \begin{pmatrix} 1 \\ -\tilde{\omega} \end{pmatrix} \right], \qquad (3.1)$$

where $L > 0$,

$$\eta_L(u) \;=\; \frac{1}{\sqrt{L}}\, \eta\!\left(\frac{u - L^2}{L} \right),$$

with $\eta \in C_0^\infty(\mathbb{R}_+)$ being a smooth cut-off function, and where $\Theta_{\tilde{n},\tilde{\omega}}(\vartheta)$ is an eigenfunction of the angular operator $\mathcal{A}$.

In the next proposition we compute E_{tot} asymptotically as $L \to \infty$.

Proposition 3.1 *For the initial data given by the wave packet (3.1),*

$$\lim_{L\to\infty} E_{\text{tot}}(\Psi_0) \;=\; \frac{1}{4\pi} \lim_{L\to\infty} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2}\, \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2}\,, \qquad (3.2)$$

where

$$\grave{\mathbf{\Psi}}_0^{\omega n} \;:=\; \begin{pmatrix} <\grave{\Psi}_{\omega n}, \Psi_0> \\ <\overline{\grave{\Psi}_{\omega n}}, \Psi_0> \end{pmatrix}. \qquad (3.3)$$

Proof. Since the wave packet is localized near infinity, it is preferable to work with the fundamental solutions $\grave{\phi}$ having the asymptotics (1.15). They are related to the functions $\phi_{1/2}$ (2.4) by

$$\begin{pmatrix} \grave{\phi} \\ \overline{\grave{\phi}} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad (3.4)$$

where

$$\mathbf{A} \;:=\; \begin{pmatrix} \alpha+\beta & i(\alpha-\beta) \\ \overline{\alpha}+\overline{\beta} & -i(\overline{\alpha}-\overline{\beta}) \end{pmatrix} .$$

In order to compute the inverse of $\mathbf{A}$, we first note that, from the asymptotics (1.14) and (1.15) of the fundamental solutions of the radial equation at infinity and at the event horizon, we obtain the following Wronskians,

$$w(\grave{\phi}, \overline{\grave{\phi}}) \;=\; -2i\omega\;, \quad w(\acute{\phi}, \overline{\acute{\phi}}) \;=\; 2i\Omega\;.$$

On the other hand, from (2.3), we know that

$$w(\grave{\phi}, \overline{\grave{\phi}}) \;=\; (\,|\alpha|^2 - |\beta|^2\,)\,w(\acute{\phi}, \overline{\acute{\phi}})\;.$$

The last two identities imply that

$$|\alpha|^2 - |\beta|^2 \;=\; -\frac{\omega}{\Omega}\;. \tag{3.5}$$

Using this relation, we obtain

$$\mathbf{B} \;:=\; \mathbf{A}^{-1} \;=\; \frac{\Omega}{2\omega} \begin{pmatrix} -\overline{\alpha}+\overline{\beta} & -\alpha+\beta \\ i(\overline{\alpha}+\overline{\beta}) & -i(\alpha+\beta) \end{pmatrix} .$$

Using again the notation (2.5), we have

$$\mathbf{\Psi}_0^{\omega n} \;=\; \overline{\mathbf{B}}\grave{\mathbf{\Psi}}_0^{\omega n}\;, \tag{3.6}$$

so we can write the total energy as

$$E_{\mathrm{tot}} \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \mathbf{B}^t\mathbf{T}\overline{\mathbf{B}}\, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2}\;.$$

The quadratic form can be written in components as

$$\langle \grave{\mathbf{\Psi}}_0^{\omega n}, \mathbf{B}^t\mathbf{T}\overline{\mathbf{B}}\, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2} \;=\; (\overline{\grave{\mathbf{\Psi}}_0^{\omega n}})_1 (\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}})_{11} (\grave{\mathbf{\Psi}}_0^{\omega n})_1 \;+\; (\overline{\grave{\mathbf{\Psi}}_0^{\omega n}})_2 (\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}})_{22} (\grave{\mathbf{\Psi}}_0^{\omega n})_2 \tag{3.7}$$

$$+(\overline{\grave{\mathbf{\Psi}}_0^{\omega n}})_1 (\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}})_{12} (\grave{\mathbf{\Psi}}_0^{\omega n})_2 + (\overline{\grave{\mathbf{\Psi}}_0^{\omega n}})_2 (\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}})_{21} (\grave{\mathbf{\Psi}}_0^{\omega n})_1\;, \tag{3.8}$$

and the matrix $\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}}$ is given by

$$\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}} \;=\; \frac{\Omega}{2\omega} \begin{pmatrix} 1 & -\overline{\alpha}/\beta \\ -\alpha/\overline{\beta} & 1 \end{pmatrix} .$$

We consider the contribution by the diagonal terms (3.7) and the off-diagonal terms (3.8) separately. Re-expressing the diagonal elements in terms of $\grave{\mathbf{\Psi}}_0^{\omega n}$ we obtain

$$(3.7) \;=\; \frac{\Omega}{2\omega}\, \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2}\;,$$

and the corresponding contribution to E_{tot} is given by (3.2).

It remains to show that the contribution of the off-diagonal terms (3.8) to E_{tot} tends to zero as $L \to \infty$. Intuitively, this can be understood from the fact that the off-diagonal terms involve oscillatory terms $e^{\pm i\omega L^2}$, whose integral becomes small for large L. In order to make the argument precise, we need to integrate by parts. We only consider the $(1,2)$ element; the argument for the $(2,1)$ element is similar. We write

$$\frac{1}{\omega\Omega}\,(\overline{\grave{\mathbf{\Psi}}_0^{\omega n}})_1(\mathbf{B}^t\mathbf{T}\overline{\mathbf{B}})_{12}(\grave{\mathbf{\Psi}}_0^{\omega n})_2 \;=\; e^{-2i\omega L^2}\,g_n(\omega)$$

with

$$g_n(\omega) \;=\; -\frac{1}{2}\,\frac{\overline{\alpha}}{\beta}\left(\frac{1}{\omega}\,<\!\Psi_0, e^{i\omega L^2}\grave{\Psi}^{\omega n}\!>\right)\left(\frac{1}{\omega}\,<\!\overline{e^{i\omega L^2}\grave{\Psi}^{\omega n}}, \Psi_0\!>\right). \tag{3.9}$$

The corresponding contribution to $E_{\rm tot}$ is obtained by integrating over ω. Note that the two brackets in (3.9) are bounded near $\omega = 0$ because the energy scalar product involves a factor of ω (see [6, Eq. (2.14)]). Furthermore, using the estimates for the Jost functions in [7, Section 3.2] and the convexity argument in [7, Section 5], we conclude that the function g_n is bounded near $\omega = 0$, and thus

$$\int_{-\delta}^{\delta}\left|e^{-2i\omega L^2}\,g_n(\omega)\right|d\omega \;\leq\; C\,\delta\,.$$

Moreover, from the estimates of the Jost solution in [7, Section 3.1] and the causality argument in [7, Section 7] we know that g is smooth, has rapid decay, and is non-zero for $\omega \in \mathbb{R}\setminus 0$. Thus we can integrate by parts,

$$\int_{\mathrm{I\!R}\setminus(-\delta,\delta)} e^{-2i\omega L^2}\,g_n(\omega)\,d\omega \;=\; \frac{1}{2iL^2}\,e^{-2i\omega L^2}\,g_n(\omega)\Big|_{-\delta}^{\delta} \;+\; \frac{1}{2iL^2}\int_{\mathrm{I\!R}\setminus(-\delta,\delta)} e^{-2i\omega L^2}\,g_n'(\omega)\,d\omega\,.$$

The boundary terms clearly tend to zero as $L \to \infty$. Hence it suffices to show that the integral on the right grows at most linearly in L. If the ω-derivative acts on the factors α, β or ω in (3.9), this integral is uniformly bounded in L. Hence it remains to consider the ω-derivatives of the energy product, like for example the term $\partial_\omega\!<\!\Psi_0, e^{i\omega L^2}\grave{\Psi}^{\omega n}\!>$. We introduce the new integration variable $v = u - L^2$ and differentiate with respect to ω. Using the asymptotics of the fundamental solutions near infinity, one sees that

$$\left|\partial_\omega\left(e^{i\omega L^2}\grave{\Psi}^{\omega n}(v + L^2)\right)\right| \;\leq\; C\,L\,. \tag{3.10}$$

with the constant C independent of $\omega \in \mathbb{R}\setminus(-\delta,\delta)$. Indeed, when we use the variable v instead of u and feed into the left-hand side of (3.10) the leading plane wave asymptotics at infinity for the fundamental solution $\grave{\Psi}^{\omega n}$, the terms quadratic in L cancel out, and the behavior of the ω derivative of the error term is controlled at infinity thanks to the Jost function estimates of Lemma 3.4 in [7]. More precisely, using the asymptotics (1.14) and (1.15), we obtain

$$\grave{\Psi}^{\omega n}(v + L^2) \;=\; e^{-i\omega(v+L^2)} + \mathcal{R}(v,\omega)\,,$$

where $\mathcal{R}(v,\omega)$ is the error term, and we therefore have

$$\partial_\omega(e^{i\omega L^2}\grave{\Psi}^{\omega n}(v + L^2)) \;=\; -ive^{-i\omega v} + e^{i\omega L^2}\partial_\omega\mathcal{R}\,.$$

The conclusion follows now from the Jost function estimates for $\grave{\phi}$ in Lemma 3.4 in [7] .

Furthermore, we note that the energy scalar product (1.20) can be written as the sum of the energy scalar product $<\!.,.\!>_0$ obtained by setting $M = 0$ in (1.20) (this is the energy scalar product in Minkowski space, expressed in oblate spheroidal coordinates) with an error term which decays like $1/r$,

$$<\Psi, \Psi'> \;=\; <\Psi, \Psi'>_0 + \int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \left\{ \frac{2Mr(r^2+a^2)}{\Delta}\, \overline{\partial_t \Phi}\, \partial_t \Phi' - 2Mr\, \overline{\partial_r \Phi}\, \partial_r \Phi' \right.$$
$$\left. - \frac{2Mra^2}{\Delta(r^2+a^2)}\, \overline{\partial_\varphi \Phi}\, \partial_\varphi \Phi' \right\} . \qquad (3.11)$$

Therefore, a short computation using (3.10) gives

$$\left| \partial_\omega <\Psi_0, e^{i\omega L^2} \grave{\Psi}^{\omega n}> \right| \;\leq\; C\, L\, .$$

∎

The result of Proposition 3.1 can be understood more directly from the following considerations, which could even be developed into an alternative proof. For large L the wave packet is localized near infinity, and thus its energy is well-approximated by the energy in Minkowski space. Considering only one angular mode and integrating out the angular variables, we thus obtain

$$E_{\text{tot}} \;=\; \int_{-\infty}^\infty \left(|\partial_t \phi_0|^2 + |\partial_r \phi_0|^2 \right) du \;+\; \mathcal{O}(L^{-1})\, .$$

Writing the solution of the one-dimensional wave equation as a Fourier integral involving left- and right-going waves,

$$\phi(t,u) \;=\; \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\omega t} \left(\hat{\phi}_L(\omega)\, e^{-i\omega u} + \hat{\phi}_R(\omega)\, e^{i\omega u} \right) d\omega\, ,$$

we can rewrite E_{tot} as

$$E_{\text{tot}} \;=\; \frac{1}{\pi} \int_{-\infty}^\infty \omega^2 \left(|\hat{\phi}_L(\omega)|^2 + |\hat{\phi}_R(\omega)|^2 \right) d\omega \;+\; \mathcal{O}(L^{-1})\, . \qquad (3.12)$$

Next, using the asymptotics (1.15) of the fundamental solutions $\grave{\phi}$ together with the asymptotic form of the energy density in (1.20), we find that

$$\hat{\phi}_L(\omega) \;=\; \frac{1}{2\omega^2} <\grave{\Psi}^{\omega n}, \Psi_0> \;+\; \mathcal{O}(L^{-1})\, , \quad \hat{\phi}_R(\omega) \;=\; \frac{1}{2\omega^2} <\overline{\grave{\Psi}^{\omega n}}, \Psi_0> \;+\; \mathcal{O}(L^{-1})\, .$$

Using this formula in (3.12), we again get the expression in Proposition 3.1.

Using the same method as in Proposition 3.1, we next compute the outgoing energy E_{out} for our wave packet.

Proposition 3.2 *For the initial data given by the wave packet (3.1),*

$$\lim_{L\to\infty} E_{\text{out}}(\Psi_0) \;=\; \frac{1}{4\pi} \lim_{L\to\infty} \sum_{n\in\mathbb{N}} \int_{-\infty}^\infty \frac{d\omega}{\omega^2} \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \mathbf{R}\, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2}\, , \qquad (3.13)$$

where $\mathbf{R}$ *is the matrix*

$$\mathbf{R} \;=\; \begin{pmatrix} \frac{|\alpha|^2}{|\beta|^2} & 0 \\ 0 & 1 \end{pmatrix} . \qquad (3.14)$$

Proof. Using (3.6), we can write the result of Proposition 2.4 as

$$E_{\rm out} \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \mathbf{B}^t\mathbf{T}\mathbf{Q}\mathbf{T}\overline{\mathbf{B}}\, \grave{\mathbf{\Psi}}_0^{\omega n}\rangle_{\mathbb{C}^2}\;.$$

The matrix $\mathbf{B}^t\mathbf{T}\mathbf{Q}\mathbf{T}\overline{\mathbf{B}}$ can be computed to be

$$\mathbf{B}^t\mathbf{T}\mathbf{Q}\mathbf{T}\overline{\mathbf{B}} \;=\; \frac{\Omega}{2\omega} \begin{pmatrix} |\alpha|^2/|\beta|^2 & -\overline{\alpha}/\beta \\ \alpha/\overline{\beta} & 1 \end{pmatrix}.$$

The diagonal terms give rise to (3.13). The off-diagonal terms, on the other hand, are exactly the same as those considered in Proposition 3.1. Thus we already know that they vanish in the limit $L \to \infty$. ■

4 Computation of Energy Gain

Combining Propositions 3.1 and 3.2, we obtain

$$\lim_{L\to\infty} \frac{E_{\rm out}}{E_{\rm tot}} \;=\; \frac{\displaystyle \lim_{L\to\infty} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \mathbf{R}(\omega,n)\, \grave{\mathbf{\Psi}}_0^{\omega n}\rangle_{\mathbb{C}^2}\, \frac{d\omega}{\omega^2}}{\displaystyle \lim_{L\to\infty} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \grave{\mathbf{\Psi}}_0^{\omega n}\rangle_{\mathbb{C}^2}\, \frac{d\omega}{\omega^2}}\;. \tag{4.1}$$

To discuss this formula, let us see how to recover Starobinsky's result (1.19). To this end, we replace the superposition in the numerator and denominator by a single wave mode with quantum numbers $\tilde{\omega} \neq 0$ and $\tilde{n}$. Then (4.1) simplifies to

$$\frac{E_{\rm out}}{E_{\rm tot}} \;=\; \frac{\langle \grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}, \mathbf{R}(\tilde{\omega},\tilde{n})\, \grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}\rangle_{\mathbb{C}^2}}{\langle \grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}, \grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}\rangle_{\mathbb{C}^2}}\;.$$

If we consider an outgoing wave at infinity, the vector $\grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}$ vanishes in the first component. According to (3.14), the above quotient gives one. This result is immediately clear, because if we take an outgoing wave near infinity, it will not interact with the black hole and simply escape to infinity. It is more interesting to consider an incoming wave at infinity. In this case, the second component of the vector $\grave{\mathbf{\Psi}}_0^{\tilde{\omega}\tilde{n}}$ vanishes, and thus

$$\frac{E_{\rm out}}{E_{\rm tot}} \;=\; \left| \frac{\alpha(\tilde{\omega})}{\beta(\tilde{\omega})} \right|^2 .$$

Expressing A and B in terms of our transmission coefficients α and β (by a straightforward calculation using (1.16, 2.3) together with (3.5)), this is in complete agreement with Starobinsky's result [11].

The remaining task is to prove that in the limit $L \to \infty$, our wave packets become "more and more localized" near $\omega = \tilde{\omega}$ and $n = \tilde{n}$.

Proof of Theorem 1.1. We consider the wave packet initial data (3.1) with $\tilde{\omega}$ and $\tilde{n}$ equal to ω and n in the statement of the theorem. We choose $c_{\rm out} = 0$ and $c_{\rm in} = 1$. From the asymptotic form of the energy density near infinity, it is obvious that the total energy of the wave packet (3.1) has a non-zero limit as Ł $\to \infty$. Furthermore, for given $\delta > 0$, the global Sobolev estimates [8] allow us to choose n_0 such that for all $L > 0$,

$$\sum_{n>n_0} \left| \int_{-\infty}^{\infty} \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2} \frac{d\omega}{\omega^2} \right| \;\leq\; \delta \,.$$

Thus we need to show that for the remaining finite number of modes,

$$\lim_{L\to\infty} \sum_{n\leq n_0} \int_{-\infty}^{\infty} \langle \grave{\mathbf{\Psi}}_0^{\omega n}, \left(\mathbf{R}(\omega, n) - \left| \frac{\alpha(\tilde{\omega})}{\beta(\tilde{\omega})} \right|^2 \right) \grave{\mathbf{\Psi}}_0^{\omega n} \rangle_{\mathbb{C}^2} \frac{d\omega}{\omega^2} \;=\; 0\,. \tag{4.2}$$

It clearly suffices to prove this for one mode.

In the case $n = \tilde{n}$, for given $\varepsilon > 0$ we choose an open neighborhood I_ε of $\tilde{\omega}$ such that

$$\frac{1}{\omega^2} \left| \left| \frac{\alpha(\omega)}{\beta(\omega)} \right|^2 - \left| \frac{\alpha(\tilde{\omega})}{\beta(\tilde{\omega})} \right|^2 \right| \;\leq\; \varepsilon \qquad \text{for all } \omega \in I_\varepsilon\,.$$

Using the asymptotic form of the energy density and of the fundamental solutions, it is obvious that

$$\begin{aligned} \lim_{L\to\infty} (\grave{\mathbf{\Psi}}_0^{\omega \tilde{n}})_2 \;&=\; 0 \quad \text{pointwise in } \mathbb{R} \\ \lim_{L\to\infty} \grave{\mathbf{\Psi}}_0^{\omega \tilde{n}} \;&=\; 0 \quad \text{pointwise in } \mathbb{R} \setminus I_\varepsilon, \end{aligned}$$

and that the functions $(\grave{\mathbf{\Psi}}_0^{\omega \tilde{n}})_2$ and $\grave{\mathbf{\Psi}}_0^{\omega \tilde{n}}$ are dominated in $\mathbb{R}$ resp. in $\mathbb{R} \setminus I_\varepsilon$ by a Schwartz function. Hence we can apply Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned} \lim_{L\to\infty} \int_{I_\varepsilon} \langle \grave{\mathbf{\Psi}}_0^{\omega \tilde{n}}, \left(\mathbf{R}(\omega, \tilde{n}) - \left| \frac{\alpha(\tilde{\omega})}{\beta(\tilde{\omega})} \right|^2 \right) \grave{\mathbf{\Psi}}_0^{\omega \tilde{n}} \rangle_{\mathbb{C}^2} \frac{d\omega}{\omega^2} \;&\leq\; \varepsilon\, E_{\text{tot}} \\ \lim_{L\to\infty} \int_{\mathbb{R}\setminus I_\varepsilon} \langle \grave{\mathbf{\Psi}}_0^{\omega \tilde{n}}, \grave{\mathbf{\Psi}}_0^{\omega \tilde{n}} \rangle_{\mathbb{C}^2} \frac{d\omega}{\omega^2} \;&=\; 0\,. \end{aligned}$$

Since ε is arbitrary, this proves (4.2) for the summand $n = \tilde{n}$.

If $n \neq \tilde{n}$, the orthogonality of the spheroidal wave functions for $\omega = \tilde{\omega}$ together with the continuity in ω allows us to choose a neighborhood I_ε of $\tilde{\omega}$ such that

$$<\Theta^{n,\omega}, \Theta^{\tilde{n},\tilde{\omega}}>_{S^2} \;\leq\; \varepsilon \qquad \text{for all } \omega \in I_\varepsilon\,.$$

Now we can use the same argument as in the case $n = \tilde{n}$ to conclude the proof of (4.2). ■

5 Absorbtion of Angular Momentum by the Black Hole

We first derive the expression for the angular momentum of the scalar wave. We recall from [6] that the Lagrangian is given by

$$\begin{aligned} \mathcal{L} \;&=\; -\Delta |\partial_r \Phi|^2 + \frac{1}{\Delta} \left| ((r^2 + a^2)\partial_t + a\partial_\varphi)\Phi \right|^2 \\ &\quad - \sin^2\vartheta\, |\partial_{\cos\vartheta} \varphi|^2 - \frac{1}{\sin^2\vartheta} \left| (a \sin^2\vartheta\, \partial_t + \partial_\varphi)\Phi \right|^2 . \end{aligned}$$

This Lagrangian is axisymmetric. Applying Noether's theorem gives rise to the following conserved quantity,

$$A[\Phi] \;=\; \int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi}\, \mathcal{A}\,,$$

where $\mathcal{A}$ is given by

$$\begin{aligned}
\mathcal{A} \;&=\; \mathrm{Re} \left(\frac{\partial \mathcal{L}}{\partial \Phi_t}\, \Phi_\varphi \right) \\
&=\; \mathrm{Re} \left\{ \frac{(r^2+a^2)^2}{\Delta}\, \overline{\partial_\varphi \Phi} \left(\partial_t \Phi + \frac{a\, \partial_\varphi \Phi}{r^2+a^2} \right) - a^2 \sin^2\vartheta\, \overline{\partial_\varphi \Phi} \left(\partial_t \Phi + \frac{\partial_\varphi \Phi}{a \sin^2\vartheta} \right) \right\}.
\end{aligned}$$

The quantity A can be interpreted as the angular momentum of the wave Φ, and $\mathcal{A}$ as the angular momentum density.

Similar to (2.8), the angular momentum absorbed by the black hole is defined by

$$A_{\mathrm{bh}} \;=\; \lim_{t\to\infty} \int_{r_1}^{2r_1} dr \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi}\, \mathcal{A}[\Phi]\,. \tag{5.1}$$

In the next proposition we compute A_{bh}, again for a fixed k-mode.

Proposition 5.1 *Choose a fixed $k \in \mathbb{Z}$ such that $\omega_0 < 0$. Then*

$$A_{\mathrm{bh}} \;=\; \frac{1}{2\pi} \sum_{n\in\mathbb{N}} \int_{-\infty}^\infty \frac{d\omega}{\omega\Omega}\, \frac{k}{\Omega} \langle \mathbf{\Psi}_0^{\omega n}, \mathbf{TPT}\, \mathbf{\Psi}_0^{\omega n} \rangle_{\mathbb{C}^2}\,.$$

Proof. Substituting the integral representation (2.1) into the definition of A_{bh}, we obtain

$$A_{\mathrm{bh}} \;=\; \lim_{t\to\infty} \frac{1}{(2\pi)^2} \int_{r_1}^{2r_1} dr \int_{-\infty}^\infty \frac{d\omega'}{\omega'\Omega'} \int_{-\infty}^\infty \frac{d\omega}{\omega\Omega}\, e^{-i(\omega-\omega')t}\, \mathcal{A}^{\omega',\omega}(r)\,,$$

where

$$\mathcal{A}^{\omega',\omega}(r) \;=\; \sum_{n,n'\in\mathbb{N}} \sum_{a,b,c,d=1}^2 t_{ab}^{\omega' n'} t_{cd}^{\omega n} <\Psi_0, \Psi_a^{\omega' n'}> <\Psi_d^{\omega n}, \Psi_0> \int_{S^2} \mathcal{A}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})\,,$$

and $\mathcal{A}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})$ denotes the bilinear form corresponding to the angular momentum density (similar to the bilinear form $\mathcal{E}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})$ appearing in the proof of Proposition 2.3). Near the event horizon, we can expand $\mathcal{A}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n})$ to obtain

$$\begin{aligned}
\frac{\Delta}{r^2+a^2}\, \mathcal{A}(\Psi_b^{\omega' n'}, \Psi_c^{\omega n}) \;&=\; \frac{1}{2}\, (r_1^2+a^2)\, k\, (\Omega+\Omega')\, \overline{\Phi_b^{\omega' n'}}\, \Phi_c^{\omega n} \;+\; \mathcal{O}(e^{\gamma u}) \\
&=\; \frac{1}{2}\, \Theta_{n'\omega'} \Theta_{n\omega}\, k\, (\Omega+\Omega')\, \overline{\phi_b^{\omega' n'}}\, \phi_c^{\omega n} \;+\; \mathcal{O}(e^{\gamma u}).
\end{aligned}$$

Now we can proceed exactly as in the proof of Proposition 2.3. ∎

Proof of Theorem 1.2. Without loss of generality we again restrict attention to the case $k > 0$, so that $\omega_0 < 0$. We set

$$\rho(\omega, n) \;=\; \frac{1}{2\pi}\, \langle \mathbf{\Psi}_0^{\omega n}, \mathbf{TPT}\, \mathbf{\Psi}_0^{\omega n} \rangle_{\mathbb{C}^2}\,.$$

The eigenvalues of the matrix **TPT** are computed to be zero and $1+|\alpha|^2/|\beta|^2$. Thus ρ is non-negative. It follows that

$$\begin{aligned} A_{\mathrm{bh}} &= \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{k}{\omega\Omega^2}\,\rho\,d\omega \;\leq\; \sum_{n\in\mathbb{N}} \int_{\omega_0}^{\infty} \frac{k}{\omega\Omega^2}\,\rho\,d\omega \\ &\leq \frac{k}{|\omega_0|} \sum_{n\in\mathbb{N}} \int_{\omega_0}^{\infty} \frac{1}{\omega\Omega}\,\rho\,d\omega \;\leq\; \frac{k}{|\omega_0|} \sum_{n\in\mathbb{N}} \int_{-\infty}^{\infty} \frac{1}{\omega\Omega}\,\rho\,d\omega \;=\; \frac{k}{|\omega_0|}\,E_{\mathrm{bh}}\,, \end{aligned}$$

and using (1.13) gives the result. ■

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References

[1] N. Andersson, P. Laguna, P. Papadopoulos, "Dynamics of scalar fields in the background of rotating black holes II: a note on superradiance," *Phys. Rev.* **D58** (1998) 087503.

[2] S. Chandrasekhar, "The Mathematical Theory of Black Holes," *Oxford University Press* (1983).

[3] D. Christodoulou, "Reversible and irreversible transformations in black hole physics," *Phys. Rev. Lett.* **25**, 1956-1957.

[4] F. Finster, N. Kamran, J. Smoller and S.-T. Yau, "The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry," *Adv. Theor. Math. Phys.* **7** (2003), 25–52.

[5] F. Finster, N. Kamran , J. Smoller and S.-T. Yau, "Decay rates and probability estimates for masssive Dirac particles in the Kerr-Newman black hole geometry," *Comm. Math. Phys.* **230** (2002), no. 2, 201–244.

[6] F. Finster, N. Kamran, J. Smoller and S.-T. Yau, "An integral spectral representation of the propagator for the wave equation in the Kerr geometry," *Comm. Math. Phys.* **260** (2005), no. 2, 257–298.

[7] F. Finster, N. Kamran, J. Smoller and S.-T. Yau, "Decay of solutions of the wave equation in the Kerr geometry," *Comm. Math. Phys.* **264** (2006), no. 2, 465–503.

[8] F. Finster, J. Smoller, "A time independent energy estimate for outgoing scalar waves in the Kerr geometry," in preparation

[9] V.P. Frolov, I.D. Novikov, "Black Hole Physics. Basic Concepts and New Developments," *Kluwer Academic Publishers Group, Dordrecht* (1998).

[10] R. Penrose, "Gravitational collapse: The role of general relativity," *Rev. del Nuovo Cimento* **1** (1969) 252–276.

[11] A.A. Starobinsky, "Amplification of waves during reflection from a black hole," *Soviet Physics JETP* **37** (1973) 28–32.

[12] S. Teukolsky, W.H. Press, "Perturbations of a rotating black hole. III. Interaction of the hole with gravitational and electromagnetic radiation," *Astrophys. J.* **193** (1974) 443–461.

[13] R. Wald, "General Relativity," *University of Chicago Press* (1984).

[14] Ya.B. Zel'dovich, "Amplification of cylindrical electromagnetic waves from a rotating body," *Soviet Physics JETP* **35** (1972) 1085–1087.

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